Functional Analysis

Part - A

1. Define complete metric space.
2. Define normed linear space
3. Define a multi-linear mapping.
4. Define functional on a ~~Hilb~~ Hilbert space.
5. If $\|x+y\| = \|x\| + \|y\|$ then prove $\|x\| \cdot \|y\| = |(x,y)|$.
6. Define Banach space with example.
7. Write the condition for two norms to be equivalent on a linear space X.
8. State closed graph theorem.
9. Define inner product space of a complex value linear space.
10. Define orthogonal and orthonormal sets in a Hilbert space H.
11. State open mapping theorem.
12. State Pythagorean theorem.
13. Define weak and strong convergence.
14. For any normed linear space X, prove that –

$$\big|\|x\| - \|y\|\big| \leq \|x-y\| \ \forall\, x,y \in X.$$

(15) Define dual space with example

(16) Define equivalent norms.

(17) Write the condition for a subset M of a Banach space $B(X, Y)$ to be uniformly bounded

(18) If $\{e_i\}$ be a orthonormal set in Hilbert Space H and x be an arbitary vector in H then write Parseval's identity.

(19) Show that for any vector x and y in a Hilbert space

$$\|x+y\|^2 - \|x-y\|^2 = 4\,\mathrm{Re}\,(x, y)$$

(20) Define the norm of continuous linear transformation.

Part - B

(1) If H is Hilbert space, prove that H is reflexiv

(2) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace $M+N$ is also closed.

(3) Show that for any vectors x, y, z in an inner product space

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2.$$

(4) Prove that if M is closed linear subspace of Hilbert subspace H, x is a vector not in M, and d is the distance from x to M, then there is a unique vector y_0 in M such that

$$\|x - y_0\| = d$$

(5) Prove that the linear space $C[-1,1]$ of all real valued continuous functions on $[-1,1]$ is not a banach space under the norm defined as:-

$$\|f\| = \int_{-1}^{1} |f(t)|\, dt$$

(6) If M is closed linear subspace of a Hilbert space H then prove that: $H = M \oplus M'$.

(7) Prove that in a Hilbert space, the inner product is jointly continuous.

(8) Let $\{e_i\}$ be an orthonormal set in Hilbert spac H, prove that:

$$\sum |(x, e_i)|^2 \leq \|x\|^2 \ \forall\ x \in H.$$

(9) A normed linear space X is a Banach space iff every absolute convergent series in X is convergent in X. Prove it.

(10) State and prove closed graph theorem for Banach spaces X and Y.

(11) If X be a normed linear space. and X_0 is non-zero vector in X then prove that

there exists a functional f_0 in X^* such that

$f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.

(12) If X be a normed linear space and $x, y \in X$ then : $\left| \|x\| - \|y\| \right| \leq \|x - y\|$ and hence show that norm is continuous.

(13) Prove that a closed convex subset M of a Hilbert space H contains a unique vector of smallest norm.

(14) Let $T: X \to Y$ be a linear transformation and if T is continuous at single point, prove that T is continuous at all points.

(15) The function space X of all bounded continuous scalar valued functions $x(t)$ defined on a set T is a Banach space under the norm $\|x\| = \sup \{|x(t)| : t \in T\}$

Prove it.

(16) State and prove projection theorem in Hilbert space

(17) State and prove Hahn-Banach Theorem.

(18) State and prove uniform boundness theorem

(19) Prove that the normed linear space $B(X, Y)$ is a Banach space if Y is a Banach space.