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Department of Mathematics and Statistics

Semester - V

Abstract Algebra and Three Dimensional Geometry

Part- A

Q.1 Define abelian group with example

Q.2 Show that the set $G = \{1, \omega, \omega^2\}$ where $1, \omega, \omega^2$ are cube root of unity is a group with ~~re~~ respect to multiplication.

Q.3 Prove that the set of n^{th} roots of unity of n-elements is a multiplicative finite abelian group.

Q.4 The inverse of an element in a group is unique.

Q.5 If $(G, *)$ be a group then $\forall$ $a \in G$ prove that.

$$(a^{-1})^{-1} = a$$

Q.6 If G is a group and $a \in G$ then show that

$$a^2 = a \text{ if and only if } a = e$$

Q.7 Define Cyclic group with example.

Q.8 What is centre of group

Q.9 Prove that set of all multiple of integers by a fixed integer 'm' is a subgroup of $(Z, +)$

Q.10 Prove that finite group of prime order does not have any proper subgroup.

Q.11 Define Isomorphism on group.

Q.12 If f be a homomorphism defined from group G to G' and e and e' are identity element of G and G' respectively then show that $f(e) = e'$

Q.13 What is Cayley's theorem

Q.14 Prove that $f: G \to G$ defined by $f(x) = x^{-1}$ $\forall$ $x \in G$ is an automorphism iff G is abelian.

Q.15 Prove that $f: (R, +) \to (R^{+}, \cdot)$ s.t $f(x) = e^{x}$ $\forall$ $x \in R$ is an isomorphism

Q.16 Define Normal subgroup

Q.17 If R is a ring and 1 is the unity, then for a, b, c, d $\in$ R then prove that

$$(-1)a = -a = a(-1)$$

Q.18 Show that set of even integers form a subring of the ring of integers

Q.19 Define Integral domain

Q.20 Prove that the field $(Q, +, \cdot)$ of rational numbers is a prime field

Q.21 Show that the set of all matrices of the form $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$, $a, b \in R$ is a ring for matrix addition and multiplication. Is it ring without zero divisor.

Q.22. Define ring with zero divisor

Q.23 Define skew field with example.

Q 24 Find the centres and radii of the following spheres: $2x^2 + 2y^2 + 2z^2 - 12x + 8y - 4z - 4 = 0$

Q. 25 Write the condition for the plane $lx + my + nz = p$ to touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Q 26 Find the two tangent plane to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$, which are parallel to the plane $2x + y - z = 4$

Q 27 If the straight line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$

intersect the curve $ax^2 + by^2 = 1$, $z = 0$ then prove that $a(\alpha n - \gamma l)^2 + b(\beta n - \gamma n)^2 = n^2$

Q 28 Find the equation to the cone generated by the lines through the origin such that their direction ratios satisfie the equation

$5l^2 - 9m^2 + n^2 - mn + nl - 5lm = 0$

Q. 29 Find the angle between the plane $x + y + z = 0$ and the cone. $\dfrac{yz}{q-r} + \dfrac{zx}{r-p} + \dfrac{xy}{p-q} = 0$

Q 30 Prove that $x^2 - y^2 + z^2 - 2x + 4y + 6z + 6 = 0$ represents a right circular cone whose vertex is the point $(1, 2, -3)$, whose axis is parallel to OY and whose semi-vertical angle is 45°

Q.31 Find the equation of a cylinder whose generating lines have the direction cosine l, m, n and which passes through the circle
$x^2 + z^2 = a^2, \quad y = 0$

Q.32 Find the equation of right circular cylinder whose axis and radius are given by.

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{1} ; 3$$

Q.33 Find the equation of right circular cone whose vertex is $(0, 0, 0)$, axis is OX and semi-vertical angle is α.

Q.34 Derived equation of Reciprocal cone

Q.35 Find the condition of tangency of plane $lx + my + nz = p$ with the ~~plane~~ sphere
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Q.36 Define tangent line and tangent plane of a sphere

Q.37 Find the equation of sphere which passes through the point $(0, -2, -4)$ and $(2, -1, -1)$ and whose the centre lies on the line $5y + 2z = 0 = 2x - 3y$.

Q.38 Define Quotient group with example.

Q.39 If H is normal subgroup of finite group G then prove $O\left(\frac{G}{H}\right) = \frac{O(G)}{O(H)}$

Q.40 Define Characteristic of a ring with example.

Part - B

Q.1 Prove that if G is an abelian group, then $(ab)^n = a^n b^n \ \forall \ a, b \in G, n \in Z$

Q.2 The order of an element of a group is always equal to the order of its inverse i.e.
$$o(a) = o(a^{-1})$$

Q.3 Prove that Every cyclic group is abelian but its converse need not be true

Q.4 The set A_n of all even permutations of degree n is a group of order $\frac{n!}{2}$ for the product of permutations.

Q.5 Find $f \cdot g$ and $g f$ for the following permutations
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$$

Q.6 If H be a subgroup of G, then prove that
(i) The identity of H is the same as that of G
(ii) The inverse of any element $a \in H$ is the same as the inverse of the element 'a' in G

Q.7 If H and K are subgroup of an abelian group G, then HK is a subgroup of G

Q.8 If G is an abelian group, then prove that $H = \{x \in G : x^2 = e\}$ is a subgroup of G

Q.9 If H be a subgroup of a group G and $a \in G, b \in G$ then prove that
$$Ha = Hb \iff ab^{-1} \in H$$

Q.10 Find all the cosets of $H = \{0, 4\}$ in the group $G = (Z_8, +_8)$

Q.11 For any fixed element $a \in G$ the mapping $f: G \longrightarrow G$ s.t. $f(x) = axa^{-1} \ \forall \ x \in G$ is automorphism

Q.12 If f is a homomorphism from a group G to G' and if the order of $a \in G$ is 'n' then order of $f(a)$ is divisor of n.

Q.13 Prove that every finite group is isomorphic to some permutation group.

Q.14 Prove that $f: G \longrightarrow G$ defined by $f(x) = x^{-1}$, $\forall \ x \in G$ is an automorphism iff G is abelian

Q.15 Prove that following function as isomorphism:
$f: (R, +) \longrightarrow (R^+, \cdot)$ s.t. $f(x) = e^x \ \forall \ x \in R$

Q.16 Prove that every subgroup of an abelian group is a normal subgroup.

Q.17 Prove that intersection of any two normal subgroups of a group is again normal subgroup.

Q.18 Every quotient group of a cyclic group is cyclic but its converse is not neccessarily true.

Q.19 Prove that Every field is an integral domain but converse is not necessarily true

Q.20 Ring $Z_p = \{0, 1, 2, \ldots\ p-1, +_p, \times_p\}$ is an integral domain iff 'p' is prime

Q.21 Prove that intersection of two subring is also a subring

Q.22 Let the set $S = \{a + b\sqrt{2} : a, b \in Q\}$ then find whether S form a subfield or not

Q.23 If R be a ring of integers S and let $S = \{mx : x \in Z\}$, m is a fixed integer then prove that S is subring of R

Q.24 A plane passes through a fixed point (a, b, c) and cuts the coordinate axis in A, B, C. show that the locus of the centre of the sphere. OABC is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

Q.25 Find the equation of the sphere having the circle $x^2+y^2+z^2=9$, $x - 2y + 2z = 5$ as a great circle.

Q.26 Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and find the point of contact

Q.27 Prove that the lines drawn from the origin so as to touch the sphere.
$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ lie on the cone
$d(x^2+y^2+z^2) = (ux+vy+wz)^2$

Q 28 Find the equation of the cone whose vertex and guiding curve are as follow:
$(1,1,1)$; $x^2+y^2+z^2=1$, $x+y+z=1$

Q 29 Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

Q 30 Find the equation of right circular cylinder. whose axis and radius are given by
$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{1} ; 3$$

Q 31 Find the equation of the enveloping cylinder of the surface. $ax^2+by^2+cz^2=1$ and whose generators are parallel to the line.
$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Part-C

Q.1 Prove that order of every element of a finite group is finite and less than or equal to the order of the group i.e. $O(a) \le O(G) \ \forall \ a \in G$

Q.2 If in a group G, $a^5 = e$ and $aba^{-1} = b^2$ $\forall \ a, b \in G$ then find the order of 'b'

Q.3 Prove necessary and sufficient condition for a non empty subset H of a group G to be a subgroup is $a \in H, \ b \in H \Rightarrow ab^{-1} \in H$

Q.4 State and prove Lagrange's theorem.

Q.5 The union of two subgroup of a group G is a subgroup iff one is contained in other. Prove this.

Q.6 If f is a homomorphism from a group G to G' and if the order of $a \in G$ is 'n' then order of $f(a)$ is divisor of 'n'

Q.7 Every Infinite cyclic group is isomorphic to the additive group of integers. Prove this

Q.8 A subgroup N of a group 'G' is normal subgroup iff $gN^{-1}g = N \ \forall \ g \in G$

Q.9 Every quotient group of an abelian group is abelian but converse is not necessarily true.

Q.10 State and prove fundamental theorem on homomorphism.

Q.11 Prove that a ring R is without zero divisor if the cancellation laws hold in R

Q.12 A finite commutative ring without zero divisor is a field

Q.13 Prove that the set J of Gaussian integers $J = \{a+ib : a, b \in Z\}$ is an integral domain with respect to addition and multiplication of complex numbers.

Q.14 Every homomorphic image of a commutative ring is a commutative ring. prove this

Q.15 If $(Z, +, \cdot)$ and $(E, +, \cdot)$ are the rings of integers and even integers respectively then prove that $\phi : Z \longrightarrow E$ defined as $\phi(a) = 2a \;\forall\; a \in Z$ is not a ring homomorphism

Q.16 Prove that if F be a homomorphism of a ring R into a ring R' with kernel K, then F will be an isomorphism if $K = \{0\}$

Q.17 If a sphere passes through origin $(0,0,0)$ meets the coordinate axis in A, B, C and whose radius is constant $2k$ then prove that the locus of centroid of tetrahedron OABC is $x^2+y^2+z^2 = k^2$

Q.18 Find the equations of the tangent planes to the sphere. $x^2+y^2+z^2 = 16$ which passes through the line $x+y = 5$, $x-2z = 7$

Q.19 Two cones with a common vertex pass through the curve $z^2 = 4ax$, $y=0$ and $z^2 = 4by$, $x=0$, The plane $z=0$ meets them in two conics which intersect in four concyclic points. show that the vertex lies on the surface. $z^2\left(\frac{x}{a}+\frac{y}{b}\right) = 4(x^2+y^2)$

Q.20 Find the equation of a right circular cylinder whose guiding circle passes through the point. $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$