B.Sc part-III - mathematics

paper-II - complex analysis

1. a) a stereographic projection projects circles into circles or straight lines.

b) If $f(z)$, is an analytic function of $z$, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Re\, f(z)|^2 = 2|f'(z)|^2$$

2. a) State and prove Heine-Borel theorem in complex plane.

b) Example show that the function $f$ defines by $f(z)=e^{-z^{-4}}$, $z\neq 0$ ; $f(0)=0$ is not analytic at $z=0$ although cauchy-Riemann equations are satisfied

3. a) A necessary and sufficient condition for a function $f(z)$ to possess an indefinite integers a simply connected domain $G$ is that the function is analytic in $G$. Further any two indefinite into differ by a constant.

b) The derivation of an analytic function is it self an analytic function.

4. a) State and prove Cauchy-Hadamard theorem.

b) Test for uniform convergence of series in the indicated region.

find the value of

$$\int_{|c|=1} \frac{\sin^6 z}{[z-(\pi/6)]^3} dz$$

b) Let $f(z)$ be a single valued analytic function in a simple connected domain $G$ if $a, b \in G$ then $\int_a^b f(z)\, dz = \phi(b) - \phi(a)$, where $\phi(z)$ is an indefinite integral of $f(z)$.

6. a) Suppose that we have obtained in any manner or as the definition of $f(z)$ the formula.

$$f(z) = \sum_{x \to \infty}^{0} A_n (z-z_0)^n, \ (R_2 < |z-z_0| < R_1)$$

then the series is necessarily identical with the Laurent's series for $f(z)$.

b) prove that :

$$\cos h\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n})$$

where

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \cos n\theta \cosh (2\cos\theta)\, d\theta$$

7. a) state and prove Casorati - Weierstrass theorem.

b) find the residues of

$$\frac{z^2}{(z-1)(z-2)(z-3)}$$

at $z = 1, 2, 3$ and infinity and show that their sum is zero.

8. a) Every polynomial of degree $n$ has exactly $n$-zeros.

b) find out the zeros and discuss the nature of singularities of

$$f(z) = \frac{z-2}{z^2} \sin \frac{1}{z-1}$$

9. a) In the transformation $z = \frac{i-w}{1+w}$, show that the positive half of the $w$-plane given by $r \geq 0$ corresponds to the circle $|z| \leq 1$ in the $z$-plane

b) find a bilinear transformation that maps the points $z = 2, i, -2$ into $w = 1, i, -1$ respectively.

10. a) Apply the calculus of residues to prove that

$$\int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2}, m > 0$$

b) evaluate $\int_0^\infty \frac{dx}{x^4 + a^4}, a > 0$

11. a) show that polar form of cauchy Riemann equation is $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$, where $f(z) = u + iv$ is an analytic function.

b) prove that $f(z) = \bar{z}$ is not differentiable at any point.

12. a) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$. Show that $f(z)$ subjects to condition $f\left(\frac{\pi}{2}\right) = 0$ is $\frac{1}{2} - \frac{1}{2}\cot\left(\frac{z}{2}\right)$

b) Show that continuity is a necessary but not a sufficient condition for the existence of a finite derivation.

13. a) State and prove Cauchy's Integral Theorem

b) If a function $f(z)$ is analytic within and on a simple closed contour $c$. Then prove that its derivative at a point $z_0$ inside $c$ is given by

$$f'(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z - z_0)^2}\, dz$$

14) a) State and prove Liouville's theorem.

b) evaluate :- $\dfrac{1}{2\pi i} \oint_c \dfrac{e^{n}}{(z^2-1)^2}\, dz, t > 0$

where $c$ is circle $|z| = 3$

15) a) find the Taylor's and Laurent's series which represent the function $\dfrac{z^2 - 1}{(z+2)(z+3)}$ the regions.

i) $|z| < 2$ ii) $2 < |z| < 3$ iii) $|z| > 3$

b) find the domain of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n$$

16. a) State and prove maximum modulus theorem.

b) Show that two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have same radius of convergence

17) a) If $f(z)$ has an isolated singularity at $z=z_0$ and is bounded in some deleted neighbourhood of $z_0$, then show that $z_0$ is removable singularity.

b) ~~If~~ State and prove Cauchy's Residue theorem.

18) a) If $f(z)$ and $g(z)$ are analytic function in a domain $G$ and $f(z) = g(z)$ on a subset of $G$ which has limit points in $G$. Then show that if for each $z \in D$, $f(z) = g(z)$ in whole of $G$

b) find the residue of $\dfrac{z^2}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ and infinity and show that their sum is zero.

19. a) What is a necessary condition for $w = f(z)$ to represent a conformal mapping?

b) Define analytic continuation. Show the power series $1 + z + z^2 + z^4 + z^8 + \cdots$ can not be continued analytically beyond $|z| = 1$

20. a) Apply the calculus of residue to prove:

$$\int_0^\infty \frac{\sin \pi x}{x(1-x^2)}\, dx = \pi$$

b) Find a bilinear transformation that maps the points $z = \infty, i, 0$ into the points $w = 0, i$ and $\infty$.