B.Sc (part-III) - Mathematics

paper-1 - Abstract algebra

1. a) prove that the residue classes modulo form a group with respect to addition of residue classes.

   b) If the order of an elements a of a group G is n, then prove that the order of a^p is also a provided p and n are relatively prime.

2. a) prove that every infinite cyclic group is isomorphism to the additive group of integers.

   b) State and prove the lagrange's theorem.

3. a) Define order of an element of a group. prove that the order of each element of a finite group is finite and less than or equal to the order of the group.

   b) Define even and odd permutations. prove that the set of all even permutations of degree n is a group of order $\lfloor n/2$

4. a) Define a sub group. If H and K are any two subgroups of a group G, then prove that HK is a subgroup of G if and only if $HK = KH$

b) Define a cyclic group. If a cyclic group G is generated by a set of elements a of order n, then prove that a^m is also a generator. if and only if m and n are relatively prime i.e $(m, n) = 1$

5. a) prove that a non-empty set G with a binary operation is a group if and only if :-

i) $a \times (b \times c) = (a \times b) \times c$, $\forall a, b, c \in G$

ii) $\forall a, b \in G$ the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solution in G.

6. prove that a semigroup G is a group if for all element $a, b \in G$ the equations $ax = b$ and $ya = b$ have unique solution in G

7. a) Define order of an element in a group. If the order of an element a of a group G is n, then show that the order of a^p is also n provided p and n are relatively prime.

b) prove that the set A_n of all even permutations of degree n is a group of order $\frac{n!}{2}$

8 a) prove that the union of two subgroups is a subgroup iff one is contained in the other.

b) Define cyclic group. prove that every subgro[up] of a cyclic group is cyclic

9. a) Define group. If G is a group such that $(ab)^n = a^n b^n$ for three consecutive integers a for all a, b and c, show that is abelian

b) Define even and odd permutation.

If $\sigma = \{1, 2, 7, 6, 3, 5, 8, 4\}$,

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 3 & 8 & 7 & 6 & 2 \end{pmatrix}$$ then prove that

$$\delta \sigma \delta^{-1} = (\delta(1)\, \delta(7)\, \delta(2)\, \delta(6)\, \delta(3)\, \delta(5)\, \delta(8)\, \delta(4))$$

10) If H is a subgroup of any group G and $g \in G$ then prove that

i) $gHg^{-1} = [ghg^{-1} \mid h \in H]$ is a subgroup of G

ii) If H is finite then $O(H) = O(gHg^{-1})$

11) a) prove that homomorphism f of group G is a monomorphism if and only if kernel of $f = \{e\}$, where e is the identity of G.

b) Let G and G' be two groups. prove that a mapping $f: G \to G'$ defined by $f(x) = x^{-1}\ \forall\ x \in G$ is an automorphism if and only if G is abelian

12. a) Give an example of each of the following:
   i) A subgroup H of group G which is not normal subgroup of G
   ii) A subgroup H of non-abelian group G, which is normal subgroup of G

13. a) prove that every field is an integral domain but the converse is not necessarily true

b) Show that quotient group of a cyclic group is cyclic, but its converse is not necessarily true.

14. a) Let R be a ring and $a \in R$, then prove that the normalizer of a in R is a subring of R

b) prove that the characteristic of an integral domain is either zero or prime number.

15\. a) prove that a commutative ring with unity is a field if it is a simple ring

b) Let R be a commutative ring with unity and I be an ideal of R then prove that quotient ring R/I is also commutative ring with unity.

16\. a) Let S and T be two subspaces of the vector space V(F), then prove that $L(S \cup T) = S + T$

b) If $\{x, y, z\}$ is linearly independent subset in vector space V(F) then prove that the set $\{x+y, y+z, z+x\}$ will also be linearly independent in V(F).

17\. a) Let $V = \{(z_1, z_2) \mid z_1, z_2 \in C\}$ be the vector space over the real field R, then prove that the set $S = \{(1,0), (i,0), (0,1), (0,i)\}$ is a basis of V

b) Let U(F) and W(F) be two subspaces of vector space V(F) then prove that

$$V = U \oplus W \iff V = U + W \text{ and } U \cap W = \{0\}$$

18) a) Let $V = \{(a,b) \mid a,b \in R\}$ and F be the field of real numbers then examine whether $V(F)$ is a vector space or not for the following defined operations.

$(a,b) \oplus (c,d) = (a+c, b+d)$; $p \odot (a,b) = (p^2 a, p^2 b)$, $\forall (a,b), (c,d) \in V$ and $p \in F$

b) prove the intersection of any two subspaces of a vector space $V(F)$ is also a subspaces.

19) a) Let $(Z_6, +_6, \cdot_6)$ is a ring, where $Z_6 = \{0,1,2,3,4,5\}$ Is it an integral domain? Explain your answer

b) find the order of a group of all even permutations of degree n

20) if H is a subgroup of a group G and $T = \{x \in G \mid xH = Hx\}$ then prove that T is a subgroup of G.